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Some Theoretical Results on the Economics of Forestry

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The general question of forest management can be stated as follows. Suppose the planner of a piece of forest land obtains utility in any time period from the timber content of trees harvested in that period. If the planner wishes to maximize the discounted sum of such utilities starting from any initial forest, what pattern of planting and harvesting trees should it follow? This paper provides a systematic analysis to answer the above question. In particular, the optimal solution is related to the Faustmann periodic solution and the sustained yield solution, which are prominent in the forestry literature.

1. INTRODUCTION

A well-known problem in capital theory is to find the correct time to stop a "pure aging process". This is illustrated most often with the following "tree-cutting example". Suppose the value of timber from a tree is related to its age (a) according to a function, f, and a discount rate of $\rho > 0$ is given. At what age should a tree (or a stand of trees of the same vintage) be cut down to maximize the present value of the timber content?

The answer to this question, proposed by Jevons (1871), Wicksell (1911) and others, can be stated as follows: cut the tree at an age, at which the increase in value of timber content of the standing tree over an additional unit time-interval equals the interest that can be earned over an additional unit time interval, if the revenue from cutting the tree is invested at an interest rate ρ.

From the point of view of the economics of forestry, the above solution (and, indeed, the above question) is missing something. More precisely, the solution is based on the assumption that the land on which the stand of trees was growing is not used after the trees are cut down. Consequently, it fails to encompass an important aspect of the "forest rotation problem". Once trees are removed from a given area, the land is available for new forest growth. Clearly, the longer the felling of the existing forest is delayed, the longer it takes to acquire revenues from future harvests. The opportunity cost of utilizing the forest site for the existing stand of trees must be considered. Taking this reforestation aspect into account, Faustmann (1849) noted that the question to be asked is the following. Assuming that one had an empty tract of land and one were interested in maximizing the discounted sum of the value of timber content of trees harvested (timber being evaluated at a constant price), what pattern of planting and harvesting trees should one follow?

Faustmann (1849) suggested the following answer to this question. A stand of trees should be cut at an age at which the increase in the value of the timber content of the standing trees over an additional unit time interval equals the sum of the following two factors (i) the interest that can be earned if the revenue from cutting the trees is invested at an interest rate, ρ; (ii) the interest that can be earned on the "site value" [that is, on the present value of the stream of all future revenues on this particular site]. Following Faustmann, this model has been discussed extensively by economists like Gaffney (1960), Pearse (1967) and Scott (1972). More thorough studies on the economics of forestry are contained in Schreuder (1968), Gregory (1972) and Wan (1978). A survey of several issues in the forestry literature is contained in Samuelson (1976). A very readable updated account of the literature is contained in Dasgupta (1982).

One can pose the question of forest management in even more general terms than Faustmann did. Suppose the owner of a piece of land, or the planner of a piece of forest land, obtains utility in any time period, which is determined by the timber content of trees harvested in that period. If one wishes to maximize the discounted sum of such utilities starting from any initial forest what pattern of planting and harvesting trees should it follow?

This question is a more general one in two respects, when compared to that posed by Faustmann. First, if the utility function is linear, one obtains the objective function of Faustmann as a special case. However, one can also examine the optimal decision of a monopolistic owner of forest land, who could have a strictly concave profit function. Also, the objective function would be applicable to the case of a planner concerned with the maximization of social welfare of an economy, from a tract of its forest land, where social welfare is measured by the discounted sum of one-period utilities. (These utilities could be derived from a linear or a non-linear utility function.)

Second, it is possible under this reformulation to consider the case where the owner or planner inherits a standing forest. Depending on the "initial forest", the optimal pattern of planting and harvesting could be different (and indeed is, as we show in this paper, even if we stick to the linear utility function used by Faustmann).

Our purpose, then, in this paper is to provide a systematic analysis to answer the general question of forest management posed above. For this purpose, we set up a model of forestry in Section 2. In Section 3, we prove the existence of a stationary forest which is optimal (called an Optimal Stationary Program, or for short, an OSP). We also prove the existence of a stationary shadow price, which "supports" an OSP in the sense that at this price, the sum of utility plus (discounted) intertemporal profit is maximized at the OSP among all feasible activities. Our analysis shows that an OSP is one in which the total plot of land is split into M equal sub-plots, with one sub-plot each containing input of trees of age a ($a = 0, 1, \ldots, M-1$). In each period, trees of age M are cut down, and the sub-plot so cleared is planted with seedlings (age zero trees). It is of interest to note that the age at which trees are cut at an OSP, is the same as the age at which trees are cut in a solution to the above-mentioned Faustmann problem (this problem is precisely formulated in (3.2)). It is also of interest to note that the set of OSPs (for, there may be more than one) is invariant under a change in the utility function.

In Section 4, we consider the case of a linear utility function and show that if the plot of land is initially empty, then the following "periodic solution", suggested by Faustmann, is optimal. The whole land is planted with seedlings initially, and all seedlings are allowed to grow to an age M (where M is a solution to the Faustmann problem (3.2)), at which time the whole forest is cut down and replanted with seedlings. (This process is repeated indefinitely.) If the plot of land has initially a standing forest then we show that the following rule is optimal. Initially, cut all trees of age M or more (where, again, M is a solution to the Faustmann problem (3.2)); thereafter, cut a tree if and only if it is of age M. Note that this means that if we think of the land as divided into sub-plots, according to the age of trees standing on them, then each sub-plot follows the periodic Faustmann solution.

In Section 5, we examine the question of uniqueness of an OSP in our framework. First, we provide examples in which multiple OSPs exist. Then, by assuming that there is a unique solution to the Faustmann problem (Problem (3.2)), we prove that there is only one OSP.

Finally, in Section 6, we consider the case of a strictly concave utility function (and assume that there is a unique OSP). We provide an example where starting from virgin land it is optimal to follow a periodic Faustmann solution. We also provide an example where starting from non-virgin land, it is optimal to follow a periodic solution. These examples demonstrate that there may not be any tendency of optimal programs to converge to the unique OSP asymptotically when the utility function is strictly concave. In fact, this study together with Mitra–Wan (1981) show that the asymptotic properties of optimal programs are similar when the utility function is linear, regardless of whether there is positive or zero discounting. But these properties may be quite dissimilar, when the utility function is strictly concave, depending on whether future utilities are undiscounted (in which case we have the "turnpike property", with the unique OSP as the "turnpike"), or positively discounted (in which case, a "turnpike property" need not hold, and periodic optimal solutions are definitely possible).

2. THE MODEL

2a. *Production*

Consider a framework in which the timber content of a tree is related to the age of the tree, through a production function, f, from R_+ to R_+. Given the age of a tree (a), the timber content of the tree is given by $f(a)$, for $a \geqq 0$.

The following assumptions on f are used in the paper:

Assumption 1. $f(a)=0$ for $0 \leqq a \leqq \boldsymbol{a}$, for some $\boldsymbol{a} \geqq 1$.

Assumption 2. f is continuous for $a \geqq \boldsymbol{a}$, and there is a positive integer $N > \boldsymbol{a}$, such that (i) $f(a)$ is increasing for $\boldsymbol{a} \leqq a < N$; (ii) $f(a)$ is decreasing for $a > N$.

Assumption 3. f is concave for $a \geqq \boldsymbol{a}$.

The graph of a production function satisfying Assumptions 1–3 is represented in Figure 1(a). A graph of a production function satisfying Assumptions 1 and 2, but violating Assumption 3 is represented in Figure 1(b).

2b. *Some notation*

In specifying our notation, N will refer to the positive integer of Section 2a.

Let d denote the first unit vector, and e the $(N+1)$-th unit vector of R^{N+1}, i.e. $d=(1,0,\ldots,0)$, $e=(0,0,\ldots,1)$ in R^{N+1}. Let μ and ν be the sum vectors in R^N and R^{N+1} respectively. Let I_N denote the $N \times N$ identity matrix. Define a $(N+1)\times(N+1)$ matrix

$$A=\begin{bmatrix} 0 & 1 \\ I_N & 0 \end{bmatrix}.$$

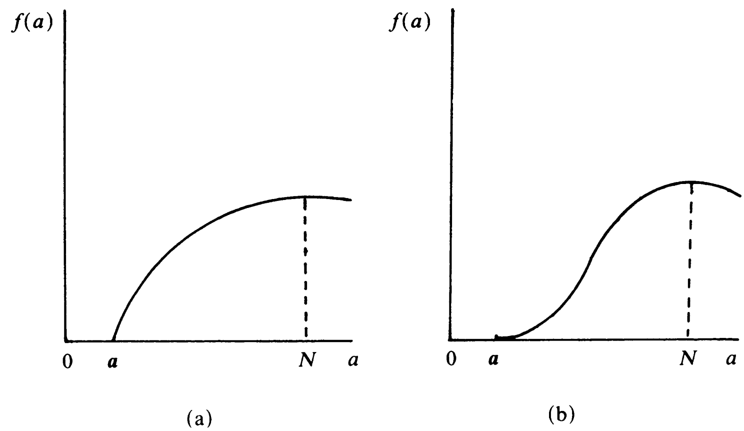

FIGURE 1

Define a $N \times (N+1)$ matrix B by

$$B = [0 \quad I_N].$$

Define a set D as follows: $D = [x \text{ in } R_+^{N+1}\text{: } \nu x = 1,\ ex = 0]$. Define a set E as follows: $E = [(x, y) \text{ in } D \times R_+^{N+1}\text{: } y = Ax]$. Note that for (x, y) in E, $\nu y = 1$, and $dy = 0$. Finally, define a set F as follows: $F = [(x, z) \text{ in } D \times D\text{: } B(Ax - z) \geqq 0]$. Note that if (x, z) is in F, then $\mu B(Ax - z) = dz$.

2c. *Programs*

Before providing rigorous definitions, we first provide an informal discussion of feasible programs. This might be helpful, since the mechanics of the model are rather simple, while this information, when written in the compact mathematical notation introduced above, may not appear to be so, at least at first glance.

Let $x_t = [x_t(0), \ldots, x_t(N)]$; then $x_t(a)$, for $a = 0, 1, \ldots, N$, is the land occupied by *input* of trees of age a, at the end of time period t. The total amount of land available for forestry in the economy is assumed to be one unit, so $\nu x_t = 1$. Also, for any reasonable objective function for the economy, trees will never be allowed to grow beyond age N; we therefore take this as a condition of feasibility itself. That is, without loss of generality, feasible programs can be restricted to those satisfying $x_t(N) = 0$, or equivalently, $ex_t = 0$. Thus, x_t belongs to the set D for each t.

Let $y_{t+1} = [y_{t+1}(0), \ldots, y_{t+1}(N)]$; then $y_{t+1}(a)$, for $a = 0, 1, \ldots, N$, is the land occupied by *output* of trees of age a, at the end of time period $(t+1)$. Since in one period a tree of age (a) becomes a tree of age $(a+1)$, so $y_{t+1}(1) = x_t(0); \ldots; y_{t+1}(N) = x_t(N-1)$. Furthermore, $y_{t+1}(0)$ is, by definition, equal to zero, that is, $dy_{t+1} = 0$. Thus, we have $y_{t+1} = Ax_t$, and (x_t, y_{t+1}) is in the set E. Note that as a consequence we have $\nu y_{t+1} = 1$, which simply reflects the fact that the total amount of land available for forestry is one unit.

At the end of time period $(t+1)$, two things are supposed to happen instantaneously, by the nature of our "point-input, point-output" framework. First, trees of different ages are cut down. Second, new seedlings (trees of age zero) are planted, in the cleared areas.

Let $x_{t+1} = [x_{t+1}(0), \ldots, x_{t+1}(N)]$; then $x_{t+1}(a)$, for $a = 0, 1, \ldots, N$, is the land occupied by input of trees of age a, at the end of time period $(t+1)$. Then, clearly, $y_{t+1}(1) \geqq x_{t+1}(1); \ldots; y_{t+1}(N) \geqq x_{t+1}(N)$. This means that $B(y_{t+1} - x_{t+1}) \geqq 0$.

Let $c_{t+1} = [c_{t+1}(1), \ldots, c_{t+1}(N)]$; then $c_{t+1}(a)$, for $a = 1, \ldots, N$, is the land released by *harvest* of trees of age a, at the end of time period $(t+1)$. Note then that $c_{t+1}(a)$ is precisely measured by $(y_{t+1}(a) - x_{t+1}(a))$ for $a = 1, \ldots, N$. Thus we have $c_{t+1} = B(y_{t+1} - x_{t+1})$. Since input of trees of age zero at the end of time period $(t+1)$ occupy the land released by all harvests, so $x_{t+1}(0) = c_{t+1}(1) + \cdots + c_{t+1}(N)$; that is $\mu c_{t+1} = dx_{t+1}$.

Keeping the above story in mind, we can now provide the formal definition of a feasible program.

A feasible program from $\boldsymbol{x}$ in D, is a sequence $\langle x_t, y_{t+1} \rangle$ satisfying

$$x_0 = \boldsymbol{x}, \qquad (x_t, y_{t+1}) \in E, \qquad B(y_{t+1} - x_{t+1}) \geqq 0 \quad \text{for } t \geqq 0. \tag{2.1}$$

Associated with a feasible program $\langle x_t, y_{t+1} \rangle$ from $\boldsymbol{x}$ in D, is a sequence $\langle c_{t+1} \rangle$ such that

$$c_{t+1} = B(y_{t+1} - x_{t+1}) \quad \text{for } t \geqq 0. \tag{2.2}$$

By the properties of sets E and F noted in Section 2b, we have

$$\nu y_{t+1} = 1, \qquad dy_{t+1} = 0, \qquad \mu c_{t+1} = dx_{t+1} \quad \text{for } t \geqq 0. \tag{2.3}$$

A feasible program $\langle x_t, y_{t+1} \rangle$ is *stationary* if $x_t = x_{t+1}$ for $t \geqq 0$. In this case, we denote the stationary levels of x_t and y_{t+1} respectively by x and y, and the stationary value of c_{t+1} by c; that is, $c = B(y - x) = B(Ax - x)$. The feasible program itself is then denoted by $\langle x, y \rangle$.

2d. *Preferences*

Preferences of the planner are represented by a utility function, u, from R_+ to R, and a discount factor, δ, in $(0, 1)$. Thus, the utility of the economy in any time period, based on the timber content of trees harvested in that period, is determined by the function, u. The discount factor, δ, depicts the time preference of the planner. The following assumptions on u are used in the paper:

Assumption 4. u is strictly increasing.

Assumption 5. u is continuous on R_+ and twice continuously differentiable on R_{++}.

Assumption 6. u is concave.

If $u(k) = mk$ for $k \geqq 0$, where $m > 0$, then the utility function is called *linear*.

Define $Q = [f(1), \ldots, f(N)]$. A feasible program $\langle x_t^*, y_{t+1}^* \rangle$ from $\boldsymbol{x}$ in D, is called *optimal* if

$$\textstyle\sum_{t=1}^{\infty} \delta^{t-1} u(Qc_t^*) \geqq \sum_{t=1}^{\infty} \delta^{t-1} u(Qc_t) \tag{2.4}$$

for every feasible program $\langle x_t, y_{t+1} \rangle$ from $\boldsymbol{x}$ in D.

We turn now to an interpretation of this definition. For a feasible program $\langle x_t, y_{t+1} \rangle$ from $\boldsymbol{x}$ in D, $c_t(a)$ for $a = 1, \ldots, N$ is the land released by harvest of trees of age a, at the end of time period t. Assuming that the trees on a plot of land are proportional to the amount of land (the factor of proportionality being unity by suitable choice of units in which the number of trees are measured), the timber content obtained by harvest at the end of time period t is given by $[f(1)c_t(1)+\cdots+f(N)c_t(N)]$, or equivalently by Qc_t. The function, u, then measures the utility obtained from this timber content at the end of time period t, $u(Qc_t)$. Implicitly, costs of planting and harvesting trees are being assumed to be zero, so that these do not enter as arguments in the utility function. If the utilities obtained in successive periods are discounted at the discount factor, δ, then one can define an optimal program $\langle x_t^*, y_{t+1}^*\rangle$ to be one which maximizes the sum of such discounted utilities; that is, by (2.4).

3. THE EXISTENCE OF AN OPTIMAL STATIONARY PROGRAM

An Optimal Stationary Program (OSP) is a stationary program which is optimal. In this section we will establish the existence of an OSP, and simultaneously provide a stationary "price support" property of such a program. This means that we will associate with the OSP a stationary shadow price vector such that the utility plus the value of various timber stands carried over, less the value of initial timber stands is maximized at the OSP among all feasible activities. Such a property is generally proved, in the literature on optimal intertemporal allocation under positive discounting (see, for example, Sutherland (1970), Peleg and Ryder (1974)) by applying a separation theorem on convex sets in finite-dimensional Euclidean spaces. Given the structure of our framework, we are able to provide a purely constructive proof, which has the advantage that we can identify immediately what the shadow prices are, in terms of the basic data (the production function, f, the utility function, u, and the discount factor, δ) of our model.

To this end, we assume Assumptions 1 and 2 and consider the function, g, defined by

$$g(a)=\delta^a f(a)/[1-\delta^a] \quad \text{for } 1\leqq a\leqq N. \tag{3.1}$$

Intuitively, $g(a)$ is the discounted "value" of an infinite sequence of planting cycles with harvesting at age a, that is,

$$g(a)=\delta^a f(a)+\delta^{2a}f(a)+\cdots.$$

Consider now the following problem (which can be referred to as the "Faustmann problem")

$$\text{maximize } g(a)$$
$$\text{subject to } a\in[1,2,\ldots,N] \tag{3.2}$$

Clearly, there is an integer, M, such that

$$\text{(i) } 1<M\leqq N$$
$$\text{(ii) } g(M)\geqq g(a) \quad \text{for } a\in[1,\ldots,N] \tag{3.3}$$

Note that $M>1$, since $g(1)=0$, while $g(N)>0$.

To prove the existence of an OSP in our framework, we need the following additional notation.

Denote $[f(M)/M]$ by β; $u'(\beta)$ by α; $P(a)=[(1-\delta^a)\delta^{M-a}f(M)/(1-\delta^M)]$ for $a=1,\ldots,N$; $P=[P(1),\ldots,P(N)]$; $q=PB$; $p=\alpha q$; $\hat{x}=[1/M,\ldots,1/M,0,0,\ldots,0]$ in D; $\theta=(1/\delta)$.

Lemma 3.1. *Under Assumptions* 1, 2, 4–6, *if* (x,z) *is in* F, *then*

$$u[QB(Ax-z)]+pz-\theta px\leqq u[\beta]+p\hat{x}-\theta p\hat{x}. \tag{3.4}$$

Proof. If (x, z) is in F, then $B(Ax-z)=[x(0)-z(1), \ldots, x(N-1)-z(N)] \geqq 0$. Now, for $a=1, \ldots, N$, we have, by definition of M,

$$Q(a)=f(a) \leqq P(a). \tag{3.5}$$

So, $Q \leqq P$. Using this information, we have

$$QB(Ax-z) \leqq PB(Ax-z)=q(Ax-z)=qAx-qz. \tag{3.6}$$

Now,

$$\begin{aligned} qA-\theta q &= [q(1), \ldots, q(N), q(0)]-[\theta q(0), \ldots, \theta q(N)] \\ &= [q(1)-\theta q(0), \ldots, q(N)-\theta q(N-1), q(0)-\theta q(N)] \\ &= [q(1), q(2)-\theta q(1), \ldots, q(N)-\theta q(N-1), -\theta q(N)]. \end{aligned}$$

Now, $q(1)=P(1)$; also, for $a=1, \ldots, N-1$,

$$\begin{aligned} q(a+1)-\theta q(a) &= P(a+1)-\theta P(a) \\ &= \frac{(1-\delta^{a+1})\delta^{M-a-1} f(M)}{(1-\delta^M)}-\frac{(1-\delta^a)\delta^{M-a} f(M)}{\delta(1-\delta^M)} \\ &= \frac{\delta^{M-a-1} f(M)}{(1-\delta^M)}[1-\delta^{a+1}-1+\delta^a] \\ &= \frac{\delta^{M-1} f(M)(1-\delta)}{(1-\delta^M)}=P(1). \end{aligned}$$

Thus, $qA-\theta q=[P(1), P(1), \ldots, P(1), -\theta q(N)]$. Hence, $qAx-\theta qx = P(1)[x(0)+\cdots+x(N-1)]-\theta q(N)x(N)=P(1)$. Using this information in (3.6),

$$QB(Ax-z)+qz-\theta qx \leqq P(1). \tag{3.7}$$

Now

$$\begin{aligned} q\hat{x}-\theta q\hat{x} &= \sum_{a=0}^{M-1} \frac{q(a)}{M}-\theta \sum_{a=0}^{M-1} \frac{q(a)}{M} \\ &= \frac{q(0)}{M}+\sum_{a=0}^{M-2} \frac{[q(a+1)-\theta q(a)]}{M}-\frac{\theta q(M-1)}{M} \\ &= \left[\frac{M-1}{M}\right] P(1)-\frac{\theta P(M-1)}{M} \\ &= \frac{\delta^{M-1} f(M)(1-\delta)}{(1-\delta^M)}\left(\frac{M-1}{M}\right)-\frac{(1-\delta^{M-1}) f(M)}{(1-\delta^M) M}. \end{aligned}$$

Hence,

$$\begin{aligned} \beta+q\hat{x}-\theta q\hat{x} &= \beta\left[1-\left\{\frac{(1-\delta^{M-1})}{(1-\delta^M)}\right\}\right]+\left\{\frac{\delta^{M-1} f(M)(1-\delta)}{(1-\delta^M)}\right\}\left\{\frac{(M-1)}{M}\right\} \\ &= f(M)\left\{\frac{\delta^{M-1}(1-\delta)}{M(1-\delta^M)}\right\}+\left\{\frac{\delta^{M-1} f(M)(1-\delta)}{(1-\delta^M) M}\right\}(M-1) \\ &= \frac{\delta^{M-1} f(M)(1-\delta)}{(1-\delta^M)}=P(1). \end{aligned}$$

Using this information in (3.7),

$$QB(Ax-z)+qz-\theta qx \leqq \beta + q\hat{x} - \theta q\hat{x}. \tag{3.8}$$

Using concavity and differentiability of u,

$$u[QB(Ax-z)] \leqq u(\beta)+\alpha[QB(Ax-z)-\beta]. \tag{3.9}$$

Combining (3.8) and (3.9), we get (3.4). ‖

Remarks

Some observations about the various magnitudes referred to in the proof of Lemma 3.1 might be appropriate. When harvesting is done at age M, $P(a)$ for $a \leqq M$ is the "value" of land with no crop; that is

$$P(a)=\delta^{-a}[\delta^M f(M)/(1-\delta^M)]-[\delta^M f(M)/(1-\delta^M)].$$

It represents the advantage of being a years nearer the harvest. If $a > M$, $P(a)$ is the "value" of a crop harvested $(a-M)$ periods ago plus the "value" of land bearing a new crop $(a-M)$ years on the way to harvest, less the "value" of land with no crop. Thus,

$$\begin{aligned} P(a) &= \delta^{M-a} f(M)+\delta^{-(a-M)}\left[\frac{\delta^M f(M)}{(1-\delta^M)}\right]-\left[\frac{\delta^M f(M)}{(1-\delta^M)}\right] \\ &= \delta^{-a}\left[\frac{\delta^M f(M)}{(1-\delta^M)}\right]-\left[\frac{\delta^M f(M)}{(1-\delta^M)}\right]. \end{aligned}$$

In view of this, $P(a+1)-\theta P(a)$ is the "value" of being one year nearer the harvest as seen at the later date. It is independent of a, since the size of the stand of timber is irrelevant; thus for $a=1,\dots,N-1$, $P(a+1)-\theta P(a)=P(1)$.

For some purposes, the following alternative price-support property will be useful. For this, however, we will use Assumption 3.

Define q' in R_+^{N+1} by: $q'(0)=0$, $q'(a)=P(a)$ for $a=1,\dots,M$, $q'(a)=f(a)$ for $a=M+1,\dots,N$. Denote $\alpha q'$ by p'. Define $P'=[q'(1),\dots,q'(N)]$.

Corollary 3.1. *Under Assumptions* 1-6, *if* (x, z) *is in* F, *then*

$$u[QB(Ax-z)]+p'z-\theta p'x \leqq u[\beta]+p'\hat{x}-\theta p'\hat{x}. \tag{3.10}$$

Proof. If (x, z) is in F, then $B(Ax-z)\geqq 0$. For $a=1,\dots,N$, we have,

$$Q(a)=f(a)\leqq P'(a). \tag{3.11}$$

So $Q\leqq P'$, and using this information, we have

$$QB(Ax-z)\leqq P'B(Ax-z)=q'(Ax-z)=q'Ax-q'z. \tag{3.12}$$

Now,

$$\begin{aligned} q'A-\theta q' &= [q'(1)-\theta q'(0),\dots,q'(N)-\theta q'(N-1), q'(0)-\theta q'(N)] \\ &= [q'(1), q'(2)-\theta q'(1),\dots,q'(N)-\theta q'(N-1), -\theta q'(N)]. \end{aligned}$$

Now, $q'(1)=P(1)$; also, for $a=1,\dots,M-1$, $q'(a+1)-\theta q'(a)=P(a+1)-\theta P(a)=P(1)$, using the proof of Lemma 3.1. For $a=M$,

$$q'(a+1)-\theta q'(a)=f(M+1)-\theta f(M)<P(M+1)-\theta P(M)$$

[since $f(M+1) < P(M+1)$, and $f(M) = P(M)] = P(1)$. For $a = M+1, \ldots, N-1$, $q'(a+1) - \theta q'(a) = f(a+1) - \theta f(a)$. Now, $f(a+1) - f(a) \leqq f(a) - f(a-1)$ using Assumption 3. Hence $f(a+1) - f(a) < \theta[f(a) - f(a-1)]$ [since $f(a) > f(a-1)$ for $a = M+1, \ldots, N-1$, and $\theta > 1$]. Thus, $f(a+1) - \theta f(a) < f(a) - \theta f(a-1)$. Hence, $f(a+1) - \theta f(a) < f(M+1) - \theta f(M)$ for $a = M+1, \ldots, N-1$. So, $q'(a+1) - \theta q'(a) < f(M+1) - \theta f(M) < P(M+1) - \theta P(M) = P(1)$, for $a = M+1, \ldots, N-1$. Thus, $q'A - \theta q' \leqq [P(1), P(1), \ldots, P(1) - \theta q'(N)]$. Now, follow exactly the proof of Lemma 3.1 replacing q everywhere by q', and p by p', to get (3.10). ||

Remark

The alternative price support property of Corollary 3.1 is used in Theorem 4.2 below to show the optimality of a certain program when the utility function is linear. The role of Assumption 3 in the corollary and the Theorem is to ensure that trees do not grow too fast in some years for $a > M$. It plays no role in Theorem 3.1, where the more general price support property of Lemma 3.1 is used to prove the existence of an OSP.

Consider the stationary program $\langle \hat{x}, \hat{y} \rangle$ given by: $\hat{x}(a) = (1/M)$ for $a = 0, 1, \ldots, M-1$; $\hat{x}(a) = 0$ for $a = M, \ldots, N$; $\hat{y} = A\hat{x}$, and $\hat{c} = B(\hat{y} - \hat{x})$. One can now use Lemma 3.1 to show that $\langle \hat{x}, \hat{y} \rangle$ is an OSP.

Theorem 3.1. *Under Assumptions* 1, 2, 4–6, $\langle \hat{x}, \hat{y} \rangle$ *is an optimal program from* $\hat{x}$.

Proof. Let $\langle x_t, y_{t+1} \rangle$ be any feasible program from x. Then, for $t \geqq 0$, (x_t, x_{t+1}) is in F. Hence, using Lemma 3.1,

$$\delta^{t-1}u(Qc_t) + \delta^{t-1}px_t - \delta^{t-2}px_{t-1} \leq \delta^{t-1}u(Q\hat{c}) + \delta^{t-1}p\hat{x} - \delta^{t-2}p\hat{x}.$$

Hence

$$\left[\textstyle\sum_{t=1}^{T} \delta^{t-1}u(Qc_t) - \sum_{t=1}^{T} \delta^{t-1}u(Q\hat{c})\right] \leqq \delta^{T-1}p(\hat{x} - x_T). \tag{3.13}$$

Since $u(0) \leqq u(Qc_t) \leqq u(f(N))$, so each sum in (3.13) converges as $T \to \infty$. Also, as $T \to \infty$, the right-hand side in (3.13) converges to zero (since x_t is in D for all t). Hence,

$$\textstyle\sum_{t=1}^{\infty} \delta^{t-1}u(Qc_t) - \sum_{t=1}^{\infty} \delta^{t-1}u(Q\hat{c}) \leqq 0 \tag{3.14}$$

which shows that $\langle \hat{x}, \hat{y} \rangle$ is an optimal program from $\hat{x}$. ||

Remark

Theorem 3.1 is not really a surprising result, in view of the general existence theorems on OSPs that are proved in the optimal growth theory literature. Note that under Assumptions 1 and 2, the production set is convex (the possible non-concavity of f having nothing to do with this feature). Furthermore, since u is concave, so in a "reduced" model with a utility function V defined on (x, z) in F, by $V(x, z) = u[QB(Ax - z)]$, V will also be concave. Thus Theorem 3.1 is related to the general existence theorems in the optimal growth literature (see, for example, McKenzie (1982)). Its novelty is in the constructive nature of the existence proof, which is possible because of the model's special structure.

It is of interest to know that the set of optimal stationary programs is invariant to a change in the utility function.

Theorem 3.2. *Under Assumptions* 1 *and* 2,

(i) *If* $\langle \bar{x}, \bar{y}\rangle$ *is an OSP for a linear utility function, then* $\langle \bar{x}, \bar{y}\rangle$ *is an OSP for every utility function satisfying Assumptions* 4–6.

(ii) *If* $\langle \bar{x}, \bar{y}\rangle$ *is an OSP for some utility function satisfying Assumptions* 4–6, *then* $\langle \bar{x}, \bar{y}\rangle$ *is an OSP for a linear utility function.*

Proof. The proof of statement (i) is quite straightforward. Consider any feasible program $\langle x_t, y_{t+1}\rangle$ from $\bar{x}$. Then for every u satisfying Assumptions 4–6, we have for $t \geqq 1$,

$$\delta^{t-1}[u(Qc_t)-u(Q\bar{c})] \leqq \delta^{t-1}u'(Q\bar{c})[Qc_t - Q\bar{c}]. \tag{3.15}$$

So

$$\sum_{t=1}^{\infty} \delta^{t-1}[u(Qc_t)-u(Q\bar{c})] \leqq \frac{u'[Q\bar{c}]}{m}\sum_{t=1}^{\infty} \delta^{t-1}m[Qc_t - Q\bar{c}]. \tag{3.16}$$

Since $\langle \bar{x}, \bar{y}\rangle$ is an OSP for a linear utility function, the right-hand side of (3.16) is non-positive. Hence the left-hand side of (3.16) is non-positive, which proves that $\langle \bar{x}, \bar{y}\rangle$ is an OSP for any u satisfying Assumptions 4–6.

The proof of (ii) is by contradiction. Suppose $\langle \bar{x}, \bar{y}\rangle$ is an OSP for some utility function, u, satisfying Assumptions 4–6, but not an OSP for a linear utility function. Then, there is some feasible program $\langle x_t, y_{t+1}\rangle$ from $\bar{x}$, and $\varepsilon > 0$, such that

$$\sum_{t=1}^{\infty} \delta^{t-1}m[Qc_t] \geqq \sum_{t=1}^{\infty} \delta^{t-1}m[Q\bar{c}] + \varepsilon \tag{3.17}$$

and

$$\sum_{t=1}^{\infty} \delta^{t-1}u[Qc_t] \leqq \sum_{t=1}^{\infty} \delta^{t-1}u[Q\bar{c}]. \tag{3.18}$$

Note that $[-u''(k)]$ is continuous on $[\frac{1}{2}Q\bar{c}, f(N)]$. Hence, there is a number, U, such that $[-u''(k)] \leqq U$ for k in $[\frac{1}{2}Q\bar{c}, f(N)]$. Choose $0 < \lambda < \frac{1}{2}$, such that $[\lambda m f(N)^2 U/u'(Q\bar{c}) \times (1-\delta)] < \varepsilon$. Define $x'_t = \lambda x_t + (1-\lambda)\bar{x}$, $y'_{t+1} = Ax'_t$ for $t \geqq 0$. Then $\langle x'_t, y'_{t+1}\rangle$ is a feasible program from $\bar{x}$, and $c'_t = \lambda c_t + (1-\lambda)\bar{c}$ for $t \geqq 1$. Note that $c'_t \geqq (1-\lambda)\bar{c} \geqq \frac{1}{2}\bar{c}$ for $t \geqq 1$.

Now, for $t \geqq 1$, we have by Taylor's expansion, some ξ_t between Qc'_t and $Q\bar{c}$ such that

$$u(Qc'_t) - u(Q\bar{c}) = u'(Q\bar{c})(Qc'_t - Q\bar{c}) + \tfrac{1}{2}u''(\xi_t)(Qc'_t - Q\bar{c})^2$$

or

$$\tfrac{1}{2}[-u''(\xi_t)](Qc'_t - Q\bar{c})^2 + [u(Qc'_t) - u(Q\bar{c})] = u'(Q\bar{c})(Qc'_t - Q\bar{c})$$

$$Qc'_t - Q\bar{c} = \lambda Qc_t + (1-\lambda)Q\bar{c} - Q\bar{c} = \lambda[Qc_t - Q\bar{c}]$$

$$(Qc'_t - Q\bar{c})^2 = \lambda^2(Qc_t - Q\bar{c})^2 \leqq \lambda^2 2f(N)^2.$$

Also since $c'_t \geqq \frac{1}{2}\bar{c}$ and $\bar{c} \geqq \frac{1}{2}\bar{c}$, so $\xi_t \geqq \frac{1}{2}Q\bar{c}$; clearly, $\xi_t \leqq f(N)$. So $[-u''(\xi_t)] \leqq U$. Thus, we have

$$\lambda^2 f(N)^2 U + [u(Qc'_t) - u(Q\bar{c})] \geqq u'(Q\bar{c})(Qc'_t - Q\bar{c})$$

So

$$\frac{\lambda^2 f(N)^2 U}{(1-\delta)} + \sum_{t=1}^{\infty} \delta^{t-1}[u(Qc'_t) - u(Q\bar{c})] \geqq \frac{u'(Q\bar{c})}{m}\sum_{t=1}^{\infty} \delta^{t-1}m[Qc'_t - Q\bar{c}]. \tag{3.19}$$

Since $\langle \bar{x}, \bar{y}\rangle$ is an OSP for the utility function u, so

$$\sum_{t=1}^{\infty} \delta^{t-1}[u(Qc'_t) - u(Q\bar{c})] \leqq 0. \tag{3.20}$$

Using (3.20), and $(Qc'_t - Q\bar{c}) = \lambda[Qc_t - Q\bar{c}]$ for $t \geqq 1$ in (3.19), we have

$$\frac{\lambda^2 f(N)^2 U}{(1-\delta)} \geqq \frac{u'(Q\bar{c})\lambda}{m} \sum_{t=1}^{\infty} \delta^{t-1} m[Qc_t - Q\bar{c}]$$
$$\geqq \frac{u'(Q\bar{c})\lambda\varepsilon}{m}.$$

Thus

$$\frac{\lambda f(N)^2 mU}{(1-\delta)u'(Q\bar{c})} \geqq \varepsilon. \tag{3.21}$$

But this contradicts the choice of λ. ||

Remark

The result of Theorem 3.2 is similar to that obtained in *aggregative* optimal growth models. In the general theory of optimal growth, where utility is defined as $V(x, z)$ on beginning and end of period stocks which are technologically compatible (see McKenzie (1982)) it is known that under some conditions if $x_t = \bar{x}$ for $t \geqq 0$ is an OSP then $V(\bar{x}, \bar{x}) \geqq V(x, z)$ for all technologically compatible (x, z) satisfying $\delta z - x \geqq (\delta - 1)\bar{x}$. But, in our model, $V(x, z) = u(QB(Ax - z))$, and since u is increasing, the maximum of V is attained at a maximum of $QB(Ax - z)$, irrespective of the precise form of u. This is the basic content of Theorem 3.2.

4. LINEAR UTILITY FUNCTION AND THE FAUSTMANN SOLUTION

When the utility function is linear, a rather complete description of optimal programs can be given. Let M be a solution to problem (3.2). Then, starting from virgin land, it is optimal to implement the following periodic policy. Let all trees grow upto age M, cut all of them down, and replant the entire forest with seedlings (age zero trees); repeat this process indefinitely. This, of course, is the solution concept proposed by Faustmann (1849) (Theorem 4.1). Note that problem (3.2) may have more than one solution; in this case, for each such solution (that is, each such M) the Faustmann periodic policy is an optimal one to follow. (For examples demonstrating the possibility of multiple solutions to (3.2), see Section 5.)

If, initially, the land is not virgin, then the following policy is optimal: initially cut all trees of age at least as large as M. Thereafter, cut a tree iff it is of age M. Note that this amounts to saying that, after the initial period, each sub-plot of land (identified by the age of trees standing on it) follows a periodic Faustmann solution, with periodicity M (where M is a solution to problem (3.2)) (Theorem 4.2).

Let M be a solution to (3.2), and consider the sequence $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ given by:

$$\begin{aligned} &\tilde{x}_0 = d, \quad \tilde{x}_t = A^t d \quad \text{for } t = 1, \dots, M-1; \\ &\tilde{x}_t = \tilde{x}_{t-M} \quad \text{for } t \geqq M; \qquad \tilde{y}_{t+1} = A\tilde{x}_t \quad \text{for } t \geqq 0. \end{aligned} \tag{4.1}$$

It can be checked that $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ is a feasible program from $\boldsymbol{x} = d$.

Theorem 4.1. *Under Assumptions* 1 *and* 2, *the feasible program* $\langle \tilde{x}_t, \tilde{y}_{t+1} \rangle$ *defined by* (4.1) *is an optimal program from* $\mathbf{x} = d$, *if the utility function is linear.*

Proof. For any feasible program $\langle x_t, y_{t+1}\rangle$ from $\boldsymbol{x}=d$, we have (x_{t-1}, x_t) in F for $t \geqq 1$. So, by using Lemma 3.1, we have, for $t \geqq 1$,

$$\delta^{t-1}[Qc_t]+\delta^{t-1}qx_t-\delta^{t-2}qx_{t-1} \leqq \delta^{t-1}[Q\hat{c}]+\delta^{t-1}q\hat{x}-\delta^{t-2}q\hat{x}. \tag{4.2}$$

Using (4.2), we have for $T \geqq 2$

$$\sum_{t=1}^{T-1} \delta^{t-1}[Qc_t-Q\hat{c}] \leqq \theta[q\boldsymbol{x}-q\hat{x}]+\delta^{T-1}[q\hat{x}-qx_T]. \tag{4.3}$$

Using (4.3), and letting $T \to \infty$, we have (since $q\boldsymbol{x}=0$),

$$\sum_{t=1}^{\infty} \delta^{t-1}[Qc_t-Q\hat{c}] \leqq -\theta q\hat{x} \tag{4.4}$$

Note, next, that for $t \geqq 1$,

$$\delta^{t-1}Q\tilde{c}_t+\delta^{t-1}q\tilde{x}_t-\delta^{t-2}q\tilde{x}_{t-1}=\delta^{t-1}P(1). \tag{4.5}$$

Also, using Lemma 3.1, we know that for $t \geqq 1$,

$$\delta^{t-1}Q\hat{c}+\delta^{t-1}q\hat{x}-\delta^{t-2}q\hat{x}=\delta^{t-1}P(1). \tag{4.6}$$

Using (4.5), (4.6), we have for $T \geqq 2$

$$\sum_{t=1}^{T-1} \delta^{t-1}[Q\tilde{c}_t-Q\hat{c}]=\theta[q\boldsymbol{x}-q\hat{x}]+\delta^{T-1}[q\hat{x}-q\tilde{x}_T]. \tag{4.7}$$

Using (4.7) and letting $T \to \infty$, we have

$$\sum_{t=1}^{\infty} \delta^{t-1}[Q\tilde{c}_t-Q\hat{c}]=-\theta q\hat{x}. \tag{4.8}$$

Combining (4.4) and (4.8) yields the desired result. ||

Remark

Note that the optimal program $\langle \tilde{x}_t, \tilde{y}_{t+1}\rangle$ from $\boldsymbol{x}=d$, does not converge to the optimal stationary program $\langle \hat{x}, \hat{y}\rangle$ [which is an optimal program from $\boldsymbol{x}=\hat{x}$]. Thus, we do not have a "turnpike property" when the utility function is linear. However, it is easy to check that the optimal program $\langle \tilde{x}_t, \tilde{y}_{t+1}\rangle$ from $\boldsymbol{x}=d$ does satisfy an "average turnpike property"; that is

$$\lim_{T\to\infty}\left[\left(\sum_{t=0}^{T} \tilde{x}_t\right)/T\right]=\hat{x} \tag{4.9}$$

where $\langle \hat{x}, \hat{y}\rangle$ is an OSP, corresponding to the same solution, M, of (3.2) as was used to construct the program $\langle \tilde{x}_t, \tilde{y}_{t+1}\rangle$.

We now consider the case in which $\boldsymbol{x}$ is an arbitrary vector in D. Let M be a solution to (3.2), and w be the M-th unit vector in R^N. Define an $(N+1)\times(N+1)$ matrix C as follows:

$$C=\begin{bmatrix} w & 1 \\ I_N & 0 \end{bmatrix}.$$

Consider the sequence $\langle x'_t, y'_{t+1}\rangle$ given by:

$$\begin{aligned} &x'_0=\boldsymbol{x}, \qquad x'_1(a)=\boldsymbol{x}(a-1) \quad \text{for } a=1,\ldots,M-1 \\ &x'_1(a)=0 \quad \text{for } a \geqq M, \qquad x'_1(0)=\sum_{a=M}^{N} \boldsymbol{x}(a-1) \\ &x'_t=[CA]^t x'_1 \quad \text{for } t=2,\ldots,M+1 \\ &x'_t=x'_{t-M} \quad \text{for } t \geqq M+2; \qquad y'_{t+1}=Ax'_t \quad \text{for } t \geqq 0. \end{aligned} \tag{4.10}$$

It can be checked that $\langle x'_t, y'_{t+1}\rangle$ is a feasible program from x. We will show that $\langle x'_t, y'_{t+1}\rangle$ is an optimal program from x.

Theorem 4.2. *Under Assumptions* 1 *and* 2, *the feasible program* $\langle x'_t, y'_{t+1}\rangle$ *defined by* (4.10) *is an optimal program from* **x** *in D, if the utility function is linear.*

Proof. First, note that

$$u(Qc'_1)+p'x'_1-\theta p'x=p'Ax-\theta p'x \tag{4.11}$$

and for $t\geqq 2$,

$$\delta^{t-1}u(Qc'_t)+\delta^{t-1}p'x'_t-\delta^{t-2}p'x'_{t-1}=P(1)\delta^{t-1}. \tag{4.12}$$

Let $\langle x_t, y_{t+1}\rangle$ be any feasible program from x. Then,

$$u(Qc_1)+p'x_1-\theta p'x\leqq p'Ax-\theta p'x \tag{4.13}$$

and for $t\geqq 2$, by Corollary 3.1

$$\delta^{t-1}u(Qc_t)+\delta^{t-1}p'x_t-\delta^{t-2}p'x_{t-1}\leqq P(1)\delta^{t-1}. \tag{4.14}$$

Using (4.11)-(4.14), we have for $T\geqq 1$,

$$\textstyle\sum_{t=1}^{T}\delta^{t-1}[u(Qc_t)-u(Qc'_t)]\leqq\delta^{T-1}p'[x'_T-x_T]. \tag{4.15}$$

Using (4.15), and letting $T\to\infty$ yields the desired result. ||

Remarks

(i) In order to clarify the description of the optimal program (given by (4.1) and (4.10)), we present below a simple example. Suppose $M=3$ and $N=4$. Let $x=(1,0,0,0,0)$; this is the case in which initially the land is virgin. In this case, an optimal program is given by $\tilde{x}_1=(0,1,0,0,0)$, $x_2=(0,0,1,0,0)$ $x_3=x$; $x_t=x_{t-3}$ for $t>3$. This is what (4.1) describes. On the other hand, suppose $x=(\frac{1}{3},\frac{1}{6},\frac{1}{2},0,0)$. This is a case in which initially there is a standing forest on the land. Then an optimal program is given by $x'_1=(\frac{1}{2},\frac{1}{3},\frac{1}{6},0,0)$, $x'_2=(\frac{1}{6},\frac{1}{2},\frac{1}{3},0,0)$, $x'_3=x$, $x'_t=x'_{t-3}$ for $t>3$. This is what (4.10) describes.

5. THE UNIQUENESS OF AN OPTIMAL STATIONARY PROGRAM

In Section 3, we established the existence of an optimal stationary program. More precisely, we showed that if M is a solution to the maximization problem (3.2), then $\hat{x}(a)=(1/M)$ for $a=0,1,\ldots,M-1$, $\hat{x}(a)=0$ for $a=M,\ldots,N$; $\hat{y}=A\hat{x}$, constitutes an optimal stationary program. Thus, if problem (3.2) has two solutions, there will be an OSP corresponding to each of these solutions. Consequently, the problem of non-uniqueness of an OSP will definitely arise when (3.2) has multiple solutions. We provide an example of a production function which satisfies Assumptions 1 and 2, but violates Assumption 3, for which (3.2) has two solutions (Example 5.1). One might be inclined to conjecture that in the context of our model with concavity of f, the uniqueness of a solution to (3.2) can be proved. (It is known, of course, in the general theory of optimal growth, that non-uniqueness of OSPs can arise with concave utility functions and convex technology sets, when future utilities are discounted.) However, remembering that the domain of maximization is a set of integers, this assumption does not really help. We show this with a second example, in which the production function satisfies Assumptions 1-3, but (3.2) still has two solutions (Example 5.2).

The question that arises next is the following: if problem (3.2) has a unique solution, will there be only one OSP? One should notice that the answer is not immediate. However, by relying on some of the results in the previous two sections we are able to answer the question in the affirmative (Theorem 5.1).

Example 5.1

Define $\phi(a)=\frac{3}{4}[1-\delta^a]/\delta^a$ for $0\leqq a\leqq N$; let $\delta=\frac{1}{2}$. Define $\psi(a)=\delta a-a^2-(39/4)$, for $0\leqq a\leqq N$.

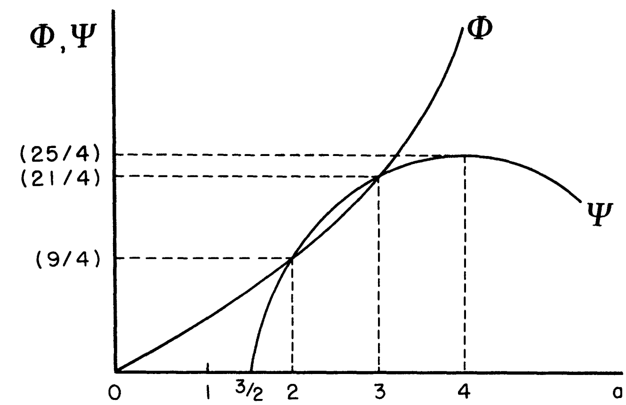

FIGURE 2

Note that ϕ is a convex function, and ψ a concave function. Also, $\phi(2)=\psi(2)=\frac{9}{4}$; $\phi(3)=\psi(3)=\frac{21}{4}$. Hence, for $0\leqq a<2$, and for $a>3$, $\psi(a)<\phi(a)$. Define

$$
\begin{aligned}
f(a)&=0 && \text{for } 0\leqq a\leqq \tfrac{3}{2}\\
&=\psi(a) && \text{for } \tfrac{3}{2}\leqq a\leqq 2\\
&=\phi(a) && \text{for } 2\leqq a\leqq 3\\
&=\psi(a) && \text{for } a\geqq 3.
\end{aligned}
$$

Then, notice that $N=4$, and we have $g(1)=0$, $g(2)=\frac{3}{4}$, $g(3)=\frac{3}{4}$, and $g(4)=\delta^4 f(4)/(1-\delta^4)<\delta^4\phi(4)/(1-\delta^4)=\frac{3}{4}$. Hence Problem (3.2) has two solutions, namely 2 and 3. Consequently, $\hat{x}=(\frac{1}{2},\frac{1}{2},0,0,0)$ and $\bar{x}=(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0)$ constitute two optimal stationary programs in this framework. Notice that f satisfies Assumptions 1 and 2 but violates Assumption 3.

Example 5.2

Define ϕ, ψ and δ as in Example 5.1.
Define

$$
\begin{aligned}
f(a)&=0 && \text{for } 0\leqq a\leqq \tfrac{3}{2}\\
&=\psi(a) && \text{for } a\geqq \tfrac{3}{2}
\end{aligned}
$$

Again, notice that $N=4$, and we have $g(1)=0, g(2)=\frac{3}{4}$, $g(3)=\frac{3}{4}$, and $g(4)<\frac{3}{4}$. Hence Problem (3.2) has two solutions, 2 and 3. Consequently, again $\hat{x}=(\frac{1}{2},\frac{1}{2},0,0,0)$ and $\bar{x}=(\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0)$ are two OSPs in this framework. Notice that f satisfies Assumptions 1-3.

To prove our uniqueness theorem, we assume that

Assumption 7. There is a unique solution, M, to Problem (3.2).

Consider now the utility function $u(k)=k$ for $k\geqq 0$. Under Assumptions 1-3, and 7, we know that $\langle \hat{x}, \hat{y}\rangle$ is an OSP, where $\hat{x}(a)=(1/M)$ for $a=0,1,\ldots,M-1$; $\hat{x}(a)=0$ for $a=M,\ldots,N$. For (x,z) in F, define $\eta(x,z)=\{\beta+p'\hat{x}-\theta p'\hat{x}\}-\{[QB(Ax-z)]+p'z-\theta p'x\}$. (To be sure, in this case, $p'=q'$.) By Corollary 3.1, we know that $\eta(x,z)\geqq 0$.

Lemma 5.1. *Under Assumptions* 1-3, *and* 7, *if* (x,x) *is in* F, *and* $\eta(x,x)=0$, *then* $x=\hat{x}$.

Proof. Using the method of proof of Corollary 3.1, we know that $\eta(x,x)=0$ implies that $QB(Ax-x)=q'(Ax-x)$, by using (3.12).

Now

$$QB=[0,f(1),f(2),\ldots,f(N)],$$
$$q'=[0,P(1),P(2),\ldots,P(M),f(M+1),\ldots,f(N)]$$

and $Ax-x=[-x_0, x_0-x_1,\ldots,x_{N-2}-x_{N-1},x_{N-1}]$. Also, $f(a)<P(a)$ for $a=1,\ldots,M-1$, so

$$x(a-1)-x(a)=0 \quad \text{for } a=1,\ldots,M-1. \tag{5.1}$$

Also, using the method of proof of Corollary 3.1 (together with that of Lemma 3.1) we know that $\eta(x,x)=0$ implies that $(q'A-\theta q')x=P(1)$. Now $q'A-\theta q'=[q'(1),\ldots,q'(N)-\theta q'(N-1),-\theta q'(N)]$ and $q'(a+1)-\theta q'(a)<P(1)$ for $a=M,\ldots,N-1$. Hence

$$x(a)=0 \quad \text{for } a=M,\ldots,N-1. \tag{5.2}$$

Combining (5.1) and (5.2), we get

$$x(a)=(1/M) \quad \text{for } a=0,\ldots,M-1. \tag{5.3}$$

Using (5.2) and (5.3), $x=\hat{x}$. ||

Theorem 5.1. *Under Assumptions* 1-7, $\langle \hat{x}, \hat{y}\rangle$ *is the unique optimal stationary program.*

Proof. By Theorem 3.1, $\langle \hat{x}, \hat{y}\rangle$ is an OSP. To prove that it is the only one, suppose, on the contrary, that there is another one (distinct from $\langle \hat{x}, \hat{y}\rangle$), call it $\langle x^*, y^*\rangle$.

By Theorem 3.2, $\langle \hat{x}, \hat{y}\rangle$ and $\langle x^*, y^*\rangle$ are OSPs for the utility function $u(k)=k$. By Theorem 4.2, the feasible program $\langle x_t', y_{t+1}'\rangle$ defined by (4.10), with x replaced by x^*, is optimal from x^*. Now, for $t=1$, using the proof of Theorem 4.2,

$$[\delta^{t-1}Qc_t'+\delta^{t-1}p'x_t'-\delta^{t-2}p'x_{t-1}']-[\delta^{t-1}Qc^*+\delta^{t-1}p'x^*-\delta^{t-2}p'x^*]\geqq 0.$$

Also, for $t>1$,

$$\begin{aligned}
&[\delta^{t-1}Qc_t'+\delta^{t-1}p'x_t'-\delta^{t-2}p'x_{t-1}']-[\delta^{t-1}Qc^*+\delta^{t-1}p'x^*-\delta^{t-2}p'x^*]\\
&=[\delta^{t-1}Q\hat{c}+\delta^{t-1}p'\hat{x}-\delta^{t-2}p'\hat{x}]\\
&\quad -\delta^{t-1}\eta(x_{t-1}',x_t')-[\delta^{t-1}Q\hat{c}+\delta^{t-1}p'\hat{x}-\delta^{t-2}p'\hat{x}]+\delta^{t-1}\eta(x^*,x^*)\\
&=\delta^{t-1}[\eta(x^*,x^*)-\eta(x_{t-1}',x_t')].
\end{aligned}$$

Hence for $T>1$,

$$\sum_{t=1}^{T} \delta^{t-1}(Qc'_t - Qc^*) = \sum_{t=2}^{T} \delta^{t-1}[\eta(x^*, x^*) - \eta(x'_{t-1}, x'_t)] + \delta^{T-1}p'x^* - \delta^{T-1}p'x'_T.$$

Hence, we have, using the fact that $\langle x^*, y^*\rangle$ is an OSP

$$0 \geqq \sum_{t=1}^{\infty} \delta^{t-1}[Qc'_t - Qc^*] \geqq \sum_{t=2}^{\infty} \delta^{t-1}[\eta(x^*, x^*) - \eta(x'_{t-1}, x'_t)].$$

Note from the proof of Theorem 4.2 that $\eta(x'_{t-1}, x'_t) = 0$ for $t \geqq 2$. Hence $\eta(x^*, x^*) = 0$. This means, by using Lemma 5.1, that $x^* = \hat{x}$. But then $\langle x^*, y^*\rangle$ is not distinct from $\langle \hat{x}, \hat{y}\rangle$, a contradiction. This establishes the theorem. ||

6. STRICTLY CONCAVE UTILITY FUNCTION AND ASYMPTOTIC PROPERTIES OF OPTIMAL PROGRAMS

In Mitra-Wan (1981), we had shown that when future utilities are undiscounted, and the utility function is strictly concave, an optimal program from any initial situation would converge to a unique optimal stationary program (called the "golden-rule"). Thus, in the undiscounted case, there was a clear qualitative difference in the asymptotic behaviour of optimal programs, depending on whether the utility function was linear or strictly concave. If initially the forestry land was virgin, optimal programs would follow a periodic Faustmann solution, when the utility function was linear, but would converge to the golden-rule solution, when the utility function was strictly concave.

When future utilities are discounted we have seen that the Faustmann periodic solution is optimal, when the utility function is linear (Theorems 4.1, 4.2). Thus, for the case of the linear utility function, the results for the undiscounted case are preserved in the discounted case.

However, for a strictly concave utility function, the results of the undiscounted case do not carry over to the discounted one. There may be two optimal stationary programs (Examples 5.1 and 5.2) and in this case, there is clearly no chance of proving a global asymptotic stability result for optimal programs.

If we suppose that problem (3.2) has a unique solution there will be a unique OSP (Theorem 5.1). We proceed in this section, under this set-up (i.e. assuming Assumption 7). Can we now show that optimal programs will converge to the unique OSP? The answer is in the negative. First, we provide an example (Example 6.1 below) in which, starting from virgin land, an optimal program follows the periodic Faustmann solution, even though the utility function is strictly concave.

A second example (Example 6.2) shows that, even if the land is not initially, virgin, an optimal program could be periodic, with the utility function strictly concave.

The examples demonstrate, we believe, ample evidence of periodic optimal programs, when the discount factor is less than one and the utility function is strictly concave. This is in contrast to our result in Mitra-Wan (1981) for the undiscounted case, where optimal programs converge to the optimal stationary program, when the utility function is strictly concave.

In a sense, our examples are not really surprising in view of the recent literature on global asymptotic stability of optimal programs when future utilities are discounted (see McKenzie (1979) for a survey), which suggests that the "turnpike property" may hold only if the discount rate is "sufficiently small" (for continuous as well as discrete-time models). However, the intertemporal model we are dealing with has much more structure than those used in this more general capital-theoretic literature. Kemp and Moore (1979), in a continuous-time model of the forestry report results of special cases in which the "turnpike property" of optimal programs will hold, when the utility function is strictly concave and future utilities are discounted. Although our framework is a discrete-time one, we feel that our examples show that "turnpike property" of optimal forestry programs cannot be a general phenomenon with a strictly concave utility function, and positive discounting.

Example 6.1

Let $f(a)=0$ for $0 \leqq a \leqq 1$, and $f(a)=30a-5a^2-25$ for $a>1$.
Note that $f'(a)=30-10a$ for $a \geqq 1$, so that $f'(3)=0$, and so $N=3$. Also $f(1)=0, f(2)=15$, $f(3)=20$. Let $\delta=\frac{1}{4}$, and $u(k)=k/[1+hk]$, where $h=\frac{1}{120}$.

It is easy to check that $\hat{x}=(\frac{1}{2}, \frac{1}{2}, 0, 0)$, $\hat{y}=A\hat{x}$, constitutes the unique OSP in this framework. This can be done by showing that

$$u(Q\hat{c})+p'\hat{x}-\theta p'\hat{x} \geqq u(Qc)+p'z-\theta p'x$$

for all (x, z) in F, with $p'=u'(\frac{15}{2})\,[0, 3, 15, 20]$, so that $\hat{x}$ is an OSP. Also, the only other stationary programs are $\langle \tilde{x}, \tilde{y} \rangle$ and $\langle \tilde{\tilde{x}}, \tilde{\tilde{y}} \rangle$ given by $\tilde{x}=(1, 0, 0, 0)$ and $\tilde{y}=A\tilde{x}$, $\tilde{\tilde{x}}=(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0)$ and $\tilde{\tilde{y}}=A\tilde{\tilde{x}}$. These are clearly not optimal programs.

We will now show that $\langle x_t, y_{t+1} \rangle$ given by $x_t=(1, 0, 0, 0)$ for t even, $x_t=(0, 1, 0, 0)$ for t odd, $y_{t+1}=Ax_t$ for $t \geqq 0$ is an optimal program from $x=(1, 0, 0, 0)$. Denote $x^*=(1, 0, 0, 0)$, $y^*=(0, 1, 0, 0)$, $c^*=(0, 0, 0)$, $z^*=(0, 1, 0, 0)$; $x^{**}=(0, 1, 0, 0)$, $y^{**}=(0, 0, 1, 0)$, $c^{**}=(0, 1, 0)$, $z^{**}=(1, 0, 0, 0)$. Define $p^*=(0, \frac{15}{4}, \frac{15}{2}, 20)$; $p^{**}=(0, \frac{15}{8}, 15, 20)$.

We will now show that for all (x, z) in F,

$$u(Qc)+p^{**}z-\theta p^* x \leqq u(Qc^*)+p^{**}z^*-\theta p^* x^* \tag{6.1}$$

and

$$u(Qc)+p^* z-\theta p^{**} x \leqq u(Qc^{**})+p^* z^{**}-\theta p^{**} x^{**}. \tag{6.2}$$

To prove (6.1), let (x, z) belong to F. Then

$$\begin{aligned} u(Qc)+p^{**}z^*-\theta p^* x &= ([15c_2+20x_2]/[1+h\{15c_2+20x_2\}])+\tfrac{15}{8}[x_0-c_1]+15[x_1-c_2] \\ &\quad -4[\tfrac{15}{4}x_1+\tfrac{15}{2}x_2] \\ &\leqq 15c_2+20x_2+\tfrac{15}{8}x_0-\tfrac{15}{8}c_1+15x_1-15c_2-15x_1-30x_2 \\ &\leqq \tfrac{15}{8}x_0-\tfrac{15}{8}c_1 \leqq \tfrac{15}{8}x_0 \leqq \tfrac{15}{8}. \end{aligned}$$

Also, note that $u(Qc^*)+p^{**}z^*-\theta p^* x^*=0+\frac{15}{8}-0=\frac{15}{8}$. This establishes (6.1).

To prove (6.2) let (x, z) belong to F. Then,

$$\begin{aligned} \pi &\equiv u(Qc)+p^* z-\theta p^{**} x \\ &= ([15c_2+20x_2]/[1+h\{15c_2+20x_2\}])+\tfrac{15}{4}(x_0-c_1)+\tfrac{15}{2}(x_1-c_2)-4[\tfrac{15}{8}x_1+15x_2] \\ &\leqq \{15c_2/[1+15hc_2]\}+20x_2+\tfrac{15}{4}x_0-\tfrac{15}{4}c_1+\tfrac{15}{2}x_1-\tfrac{15}{2}c_2-\tfrac{15}{2}x_1-60x_2 \\ &\leqq \{15c_2/[1+15hc_2]\}+\tfrac{15}{4}x_0-\tfrac{15}{2}c_2. \end{aligned}$$

Now $0 \leqq z_2=x_1-c_2$, so $c_2 \leqq x_1$, and $-c_2 \geqq -x_1$. Also $x_0+x_1+x_2+x_3=1$, so $x_0 \leqq 1-x_1 \leqq 1-c_2$. Hence

$$\begin{aligned} \pi &\leqq \{15c_2/[1+15hc_2]\}+\tfrac{15}{4}-\tfrac{15}{4}c_2-\tfrac{15}{2}c_2 \\ &= \{15c_2/[1+\tfrac{1}{8}c_2]\}-\tfrac{45}{4}c_2+\tfrac{15}{4}. \end{aligned}$$

Now, $j(k)=\{15k/[1+\frac{1}{8}k]\}-\frac{45}{4}k+\frac{15}{4}$ is increasing on the interval [0, 1], since

$$j'(k)=\{15/[1+\tfrac{1}{8}k]^2\}-\tfrac{45}{4}\geqq\{15/[\tfrac{9}{8}]^2\}-\tfrac{45}{4}=15[\tfrac{64}{81}-\tfrac{3}{4}]>0 \quad \text{for } k \text{ in } [0,1].$$

Hence, $\pi\leqq\{15/[1+\frac{1}{8}]\}-\frac{45}{4}+\frac{15}{4}=\frac{35}{6}$. Also, $u(Qc^{**})+p^*z^{**}-\theta p^{**}x^{**}=\{15/[1+\frac{1}{8}]\}+0-4(\frac{15}{8})=\frac{35}{6}$. This establishes (6.2).

To show that $\langle x_t, y_{t+1}\rangle$ is optimal from $x=(1,0,0,0)$, we define

$$\begin{aligned} p_t&=\delta^{t-1}p^* \quad \text{for } t \text{ even;} \qquad p_0=\theta p^* \\ p_t&=\delta^{t-1}p^{**} \quad \text{for } t \text{ odd.} \end{aligned} \tag{6.3}$$

Then, using (6.1), (6.2), we have for (x, z) in F, and $t\geqq 1$

$$\delta^{t-1}u(Qc)+p_tz-p_{t-1}x\leqq\delta^{t-1}u(Qc_t)+p_tx_t-p_{t-1}x_{t-1}. \tag{6.4}$$

Let $\langle x'_t, y'_{t+1}\rangle$ be any feasible program from $x=(1,0,0,0)$. Then $\langle x'_t, x'_{t+1}\rangle$ is in F for $t\geqq 0$. So using (6.4) we have

$$\textstyle\sum_{t=1}^T \delta^{t-1}[u(Qc'_t)-u(Qc_t)]\leqq p_T[x_T-x'_T]\leqq p_Tx_T.$$

Since $p_Tx_T\to 0$ as $T\to\infty$, so

$$\textstyle\sum_{t=1}^\infty \delta^{t-1}[u(Qc'_t)-u(Qc_t)]\leqq 0.$$

Hence $\langle x_t, y_{t+1}\rangle$ is optimal from x. Note that (x_t, y_{t+1}) does not converge to the optimal stationary program $(\hat{x}, \hat{y})$ as $t\to\infty$.

Example 6.2

Let f be defined as in Example 6.1. Let $\delta=\frac{1}{4}$ and $u(k)=hk-Hk^2$ for $0\leqq k\leqq 20$, with $h=\frac{4}{15}$ and $H=\frac{1}{225}=(h/60)$. Note then that $u'(k)>0$ and $u''(k)<0$ for $0\leqq k\leqq 20$.

As in Example 6.1, there is a unique OSP, given by $\hat{x}=(\frac{1}{2},\frac{1}{2},0,0)$, $\hat{y}=A\hat{x}$.

We will now show that $\langle x_t, y_{t+1}\rangle$ given by $x_t=[\frac{1}{5},\frac{4}{5},0,0]$ for t even, and $x_t=[\frac{4}{5},\frac{1}{5},0,0]$ for t odd, $y_{t+1}=Ax_t$ for $t\geqq 0$, is an optimal program from $x=[\frac{1}{5},\frac{4}{5},0,0]$.

Denote $x^*=[\frac{1}{5},\frac{4}{5},0,0]$, $y^*=[0,\frac{1}{5},\frac{4}{5},0]$, $c^*=[0,\frac{4}{5},0]$, $z^*=[\frac{4}{5},\frac{1}{5},0,0]$; $x^{**}=[\frac{4}{5},\frac{1}{5},0,0]$, $y^{**}=[0,\frac{4}{5},\frac{1}{5},0]$, $c^{**}=[0,\frac{1}{5},0]$, $z^{**}=[\frac{1}{5},\frac{4}{5},0,0]$. Define $p^*=[0,\frac{2}{5},\frac{18}{5},4]$, $p^{**}=[0,\frac{4}{5},\frac{12}{5},4]$.

We will show for all (x, z) in F,

$$u(Qc)+p^{**}z-\theta p^*x\leqq u(Qc^*)+p^{**}z^*-\theta p^*x^* \tag{6.5}$$

and

$$u(Qc)+p^*z-\theta p^{**}x\leqq u(Qc^{**})+p^*z^{**}-\theta p^{**}x^{**}. \tag{6.6}$$

To prove (6.5), let (x, z) belong to F. Then,

$$\begin{aligned} \pi&=u(Qc)+p^{**}z-\theta p^*x=h[15c_2+20x_2]-H[15c_2+20x_2]^2 \\ &\quad+p_1^{**}(x_0-c_1)+p_2^{**}(x_1-c_2)-\theta p_1^*x_1-\theta p_2^*x_2 \\ &\leqq 15hc_2+20hx_2-225Hc_2^2-225(20)^2x_2^2+p_1^{**}x_0-p_1^{**}c_1 \\ &\quad+p_2^{**}x_1-p_2^{**}c_2-\theta p_1^*x_1-\theta p_2^*x_2 \\ &\leqq 4c_2-c_2^2+p_1^{**}x_0-p_1^{**}c_1+p_2^{**}x_1-p_2^{**}c_2-\theta p_1^*x_1 \\ &\leqq 4c_2-c_2^2+p_1^{**}x_0+p_1^{**}x_1-p_2^{**}c_2 \\ &\leqq 4c_2-c_2^2+p_1^{**}-p_2^{**}c_2=4c_2-c_2^2-\tfrac{12}{5}c_2+\tfrac{4}{5}. \end{aligned}$$

Now, $j(k)=4k-k^2-\frac{12}{5}k$ attains a maximum at $k=\frac{4}{5}$. Using this,

$$\pi \leqq 4(\tfrac{4}{5})-(\tfrac{4}{5})^2-\tfrac{48}{25}+\tfrac{4}{5}=\tfrac{36}{25}.$$

Also, note that

$$u(Qc^*)+p^{**}z^*-\theta p^*x^*=4c_2^*-(c_2^*)^2+p_1^{**}\tfrac{1}{5}-\theta p_1^*\tfrac{4}{5}$$
$$=4(\tfrac{4}{5})-(\tfrac{4}{5})^2+\tfrac{1}{5}(\tfrac{4}{5})-\tfrac{32}{5}(\tfrac{1}{5})=\tfrac{36}{25}.$$

This establishes (6.5).

To prove (6.6), let (x, z) be in F. Then,

$$\pi=u(Qc)+p^*z-\theta p^{**}x=h[15c_2+20x_2]-H[15c_2+20x_2]^2$$
$$+p_1^*(x_0-c_1)+p_2^*(x_1-c_2)-\theta p_1^{**}x_1-\theta p_2^{**}x_2.$$
$$\leqq 4c_2-c_2^2+p_1^*x_0-p_1^*c_1+p_2^*x_1-p_2^*c_2-\theta p_1^{**}x_1$$
$$\leqq 4c_2-c_2^2+p_1^*x_0-p_2^*c_2+p_1^*x_1$$
$$\leqq 4c_2-c_2^2+p_1^*-p_2^*c_2=4c_2-c_2^2+\tfrac{2}{5}-\tfrac{18}{5}c_2.$$

Now, $j(k)=4k-k^2-\frac{18}{5}k$ is maximized at $k=\frac{1}{5}$. Using this,

$$\pi \leqq 4(\tfrac{1}{5})-(\tfrac{1}{5})^2+\tfrac{2}{5}-\tfrac{18}{5}(\tfrac{1}{5})$$
$$=\tfrac{11}{25}$$

Also, note that

$$u(Qc^{**})+p^*z^{**}-\theta p^{**}x^{**}=4c_2^{**}-(c_2^{**})^2+\tfrac{2}{5}(\tfrac{4}{5})-(\tfrac{4}{5})^2=\tfrac{11}{25}.$$

This establishes (6.6).

Now, using exactly the method in Example 6.1, it can be verified that $\langle x_t, y_{t+1}\rangle$ is an optimal program from $\boldsymbol{x}=[\frac{1}{5}, \frac{4}{5}, 0, 0]$, by exploiting the results (6.5) and (6.6).

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